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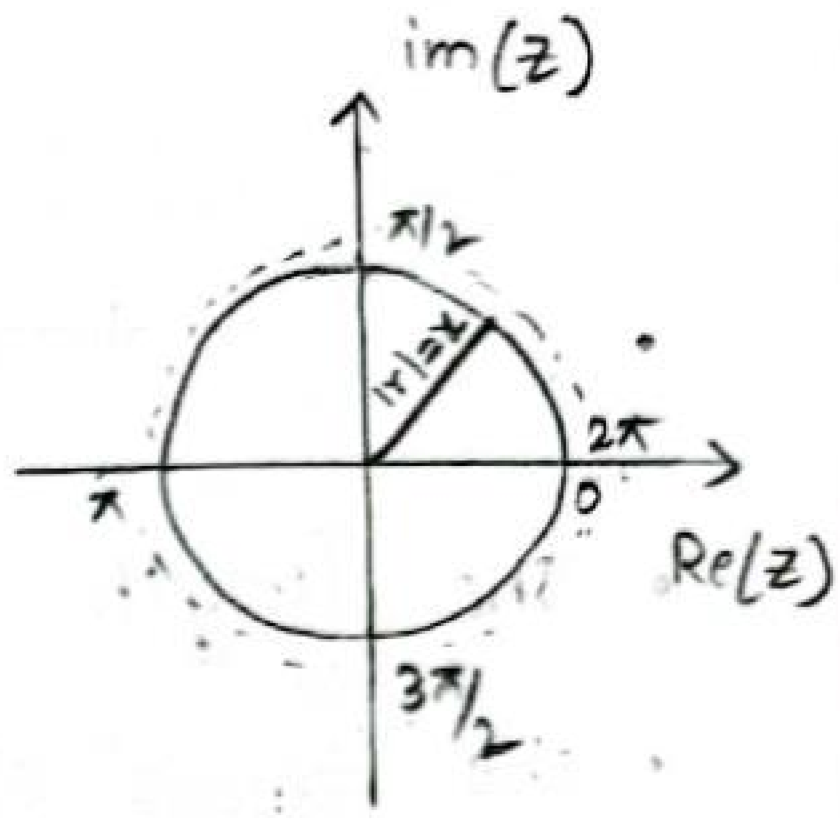
UNIT - 5

Z - TRANSFORM

↳ discrete - time system

$x(n) \xleftrightarrow{ZT} X(z)$ $\Rightarrow X(z) = z[x(n)]$

Z - plane



$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}} \quad -(1)$$

z -> complex variable

$$z = \underbrace{x + jy}_{\text{cartesian}} = \underbrace{re^{j\omega}}_{\text{Polar}}$$

$$|z| = \sqrt{x^2+y^2} = |r| \underbrace{|e^{j\omega}|}_{=1}$$

$$|z| = \sqrt{x^2+y^2} = |r|$$

$$x^2+y^2 = r^2 \quad ; \quad \omega = \tan^{-1}(y/x)$$

- $z = re^{j\omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n}$$

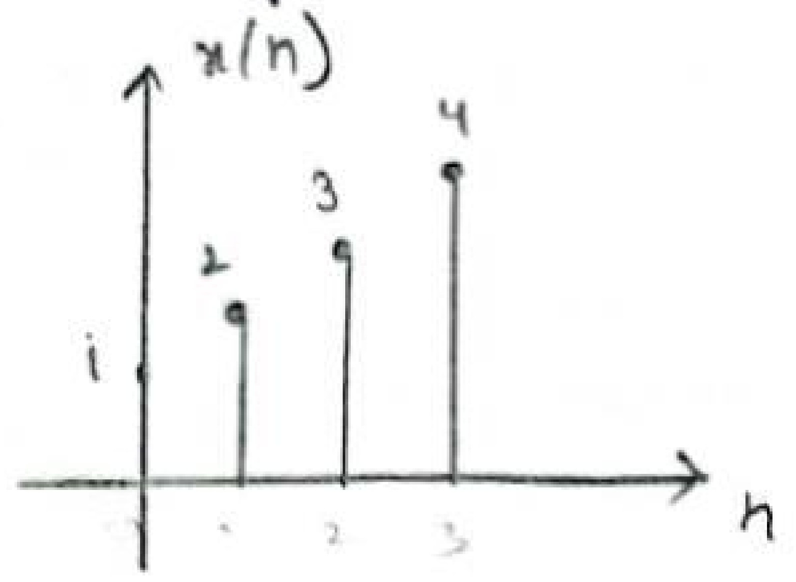
$$= \sum_{n=-\infty}^{\infty} r^{-n} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} |r^{-n} x(n)| \underbrace{|e^{-j\omega n}|}_{1}$$

$$X(z) = \sum_{n=-\infty}^{\infty} |r^{-n} x(n)| < \infty$$ $\Rightarrow$ condition for existance of z-tranform

- The range of values of 'r', over which the above series is converged is known as ROC

Eg:- 1) $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$



- The length of the signal is 4
- The above signal is finite duration causal signal.

Z-transform :- $X(z) = ZT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=0}^{3} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) + z^{-2} + x(3) z^{-3}$$

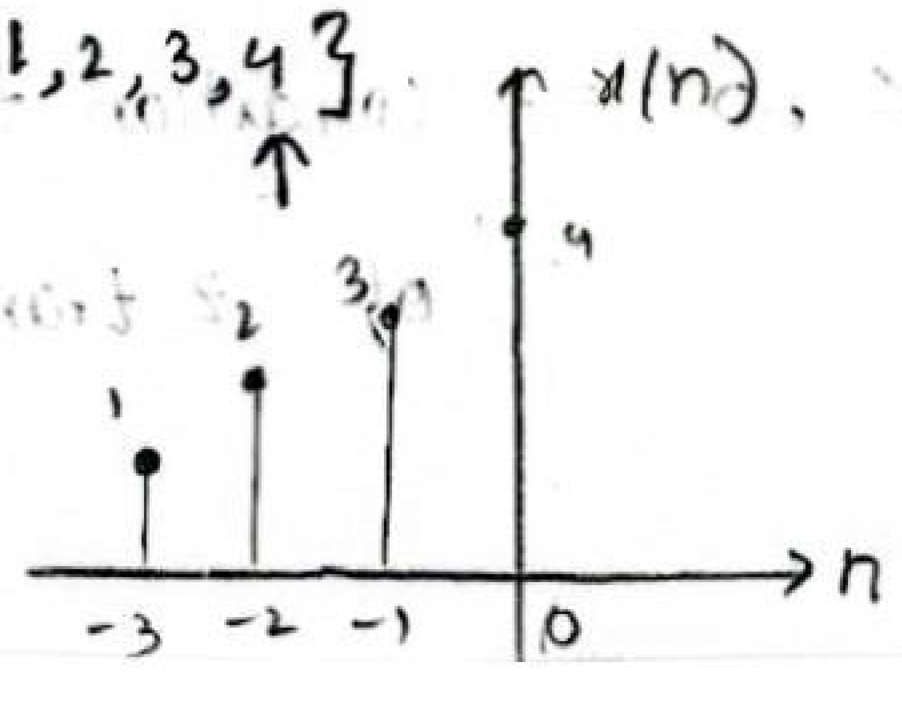
$$\boxed{X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}}$$

✓ ROC :- entire z-plane except at $|z| = r = 0$

↓ $x(0) = \frac{z^3 + 2z^2 + 3z^1 + 4}{z^3} = \infty$

- For any finite duration causal signal the ROC is always entire z-plane except at $|z| = r = 0$.

2) $x(n) = \{1, 2, 3, \underset{\uparrow}{4}\}$



- The length of the signal is 4
- The signal is finite duration anti-causal signal

z-tranform:- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=-3}^{0} x(n) z^{-n}$$

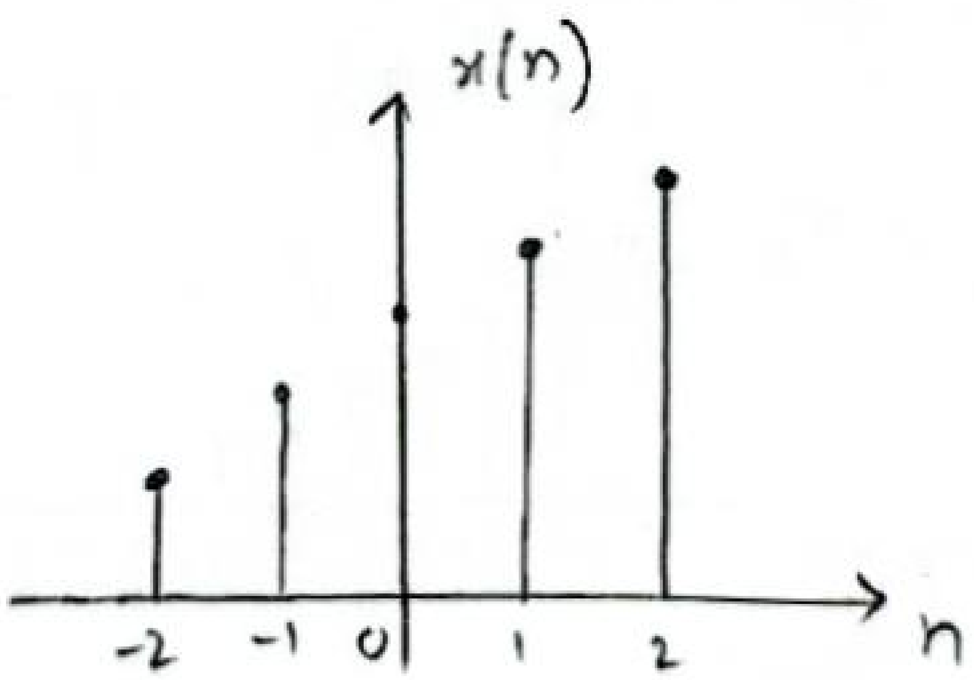
$$= x(-3) z^{+3} + x(-2) z^{+2} + x(-1) z^{+1} + x(0)$$

$$\boxed{X(z) = z^3 + 2z^2 + 3z + 4}$$

ROC :- Entire z-plane Expect at $|z| = r = \infty$

- For any finite duration anti-causal signal the ROC is entire z-plane expect at $|z| = r = \infty$

3) $x(n) = \{1, 2, \underset{\uparrow}{3}, 4, 5\}$



length = 5 , Non-causal signal

$X(z) = \sum_{n=-2}^{2} x(n) z^{-n}$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) + x(1) z^{-1} + x(2) z^{-2}$$

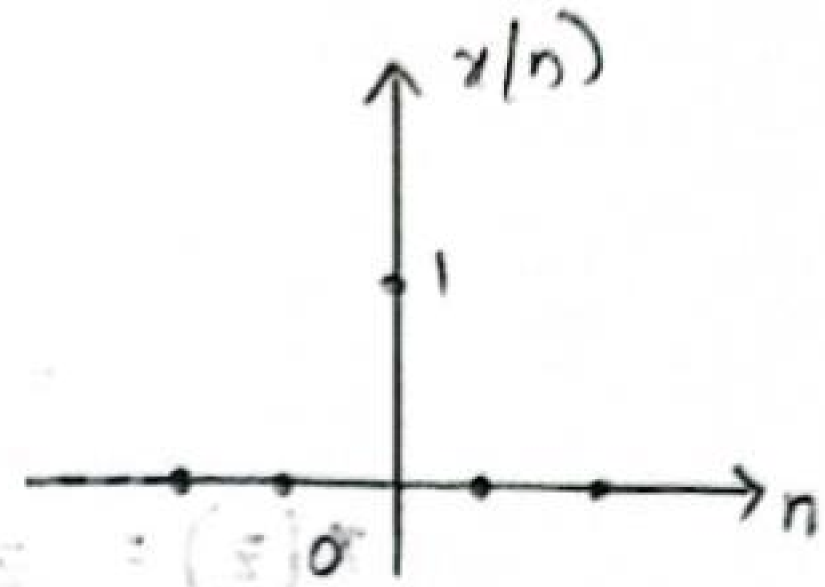
$$\boxed{X(z) = z^2 + 2z^1 + 3 + 4z^{-1} + 5z^{-2}}$$

ROC :- entire z-plane expect at $|z| = r = 0$ &

$|z| = r = \infty$

4Q, Find the z-Tranform of the signal $x(n) = \delta(n)$

$$\delta(n) = \begin{cases} 1 & ; x = 0 \\ 0 & ; else \end{cases}$$



$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

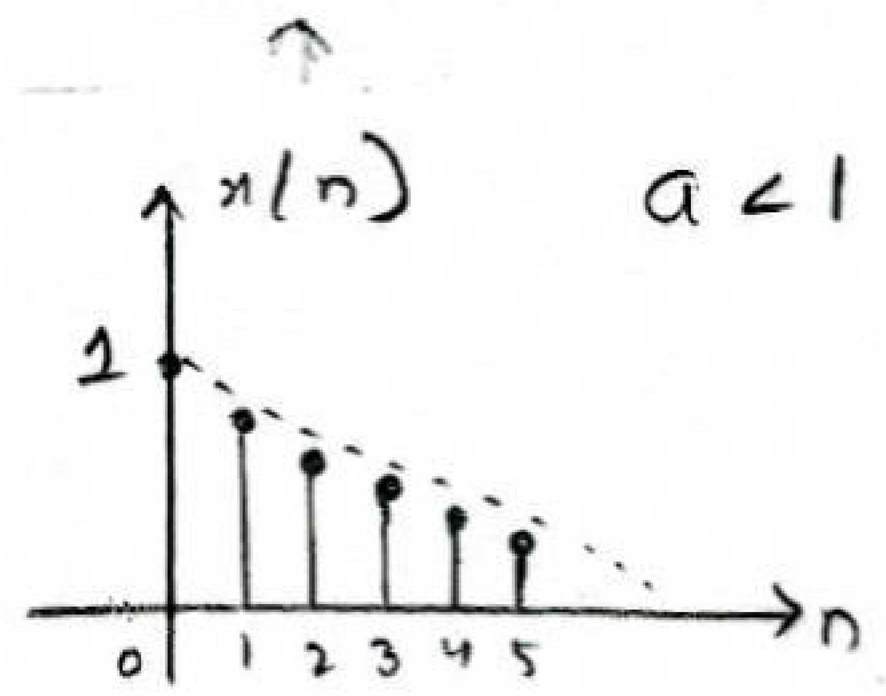
$$= \cdots \delta(-1) z^{1} + \delta(0) + \delta(1) z^{-1} + \cdots$$

$$= \cdots 0 + 1 + 0 + \cdots$$

$$\boxed{X(z) = 1}$$

ROC:- Entire z-plane

5Q, Find ZT of $x(n) = a^n u(n)$



$$X(z) = \sum_{n=-\infty}^{\infty} [a^n u(n)] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

• $a < 1 \rightarrow$ exp ↓

• $a > 1 \rightarrow$ exp ↑

$$= \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}}$$

$$\boxed{X(z) = \frac{1}{1 - a z^{-1}}}$$

$$\because \sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & |r| < 1 \\ \infty & else \end{cases}$$

$$a^n u(n) \xleftrightarrow{ZT} \frac{1}{1 - a z^{-1}}$$ [ROC

ROC:-

$$|az^{-1}| < 1$$

$$\left|\frac{a}{z}\right| < 1$$

$$\left|\frac{z}{a}\right| > 1$$

$$\boxed{|z| > |a|}$$

- For Infinite duration causal signal ROC is exterior of a circle of some radius 'r'

6) find Z.T of $x(n) = u(n)$

WKT $\quad a^n u(n) \xleftrightarrow{ZT} \dfrac{1}{1-az^{-1}}$ $\quad$ ROC : $|z| > |a|$

Put $a = 1$

$$\boxed{u(n) \xleftrightarrow{ZT} \frac{1}{1-z^{-1}}} \qquad \boxed{\text{ROC}: |z| > 1}$$

7) find Z.T of $x(n) = a^n u(-n-1)$

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{else} \end{cases}$$

$$u(-n-1) = \begin{cases} 1 & \text{for } -n-1 \geq 0 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1 & \text{for } n \leq -1 \\ 0 & \text{else} \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(-n-1)\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^n z^{-n}$$

Put $n = -m$

$$= \sum_{m=1}^{\infty} a^{-m} z^{m} = \sum_{m=1}^{\infty} (a^{-1} z)^m$$

$$X(z) = a^{-1}z + (a^{-1}z)^2 + (a^{-1}z)^3 + \cdots$$

$$= a^{-1}z \left[\underbrace{1 + a^{-1}z + (a^{-1}z)^2 + \cdots}_{G.P} \right]$$

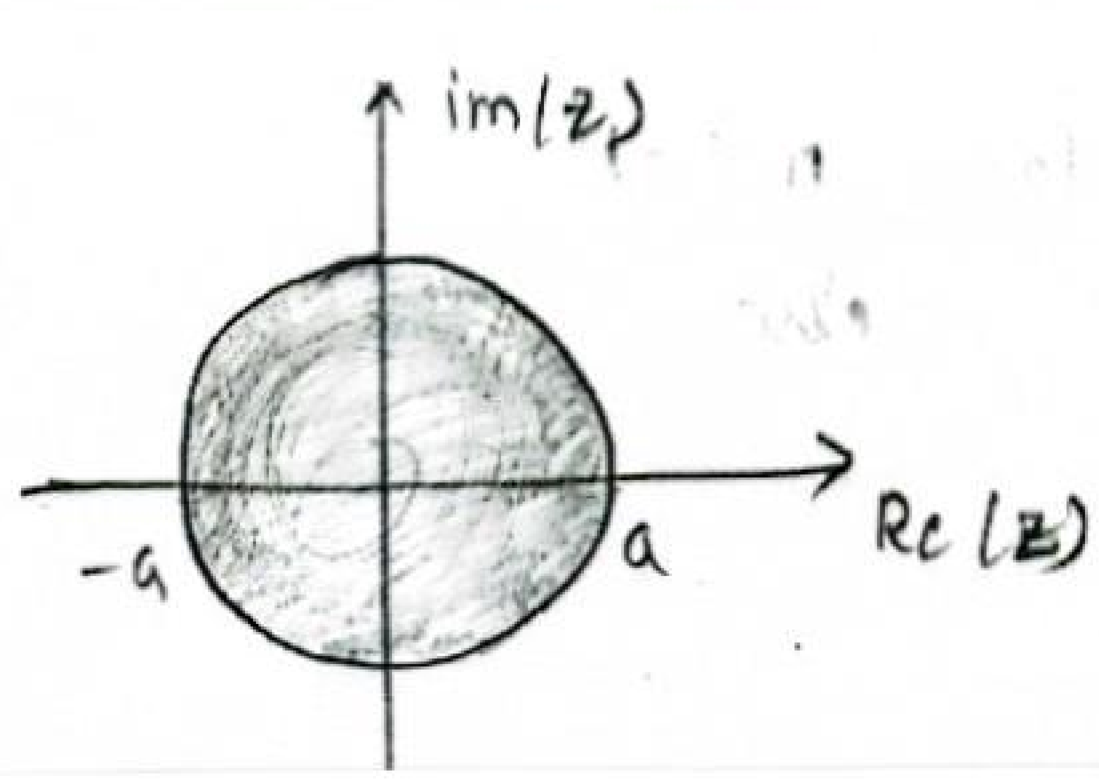
$$X(z) = \frac{a^{-1}z}{1 - a^{-1}z} \quad \text{if } |a^{-1}z| < 1$$

$$\boxed{X(z) = \frac{-1}{1 - az^{-1}}} \qquad \boxed{ROC: \; |z| < |a|}$$

$$a^n u(-n-1) \xleftrightarrow{ZT} \frac{-1}{1 - az^{-1}}$$

amplitude scaling

$$\boxed{-a^n u(-n-1) \longleftrightarrow \frac{1}{1 - az^{-1}}} \qquad \boxed{ROC: \; |z| < |a|}$$



Properties of Z.T :

1, Linearity property:

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$ ROC :- r_1 , $x_2(n) \xleftrightarrow{ZT} X_2(z)$ ROC : r_2

then $\alpha x_1(n) + \beta x_2(n) \xleftrightarrow{ZT} \alpha X_1(z) + \beta X_2(z)$ ROC : $r_1 \cap r_2$

Proof :-

$$Z[\alpha x_1(n) + \beta x_2(n)] = \sum_{n=-\infty}^{\infty} [\alpha x_1(n) + \beta x_2(n)] z^{-n}$$

$$= \alpha \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + \beta \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= \alpha X_1(z) + \beta X_2(z)$$

$$\boxed{\alpha x_1(n) + \beta x_2(n) \xleftrightarrow{ZT} \alpha X_1(z) + \beta X_2(z)}$$

$$Z[\alpha x_1(n) + \beta x_2(n)] = \alpha Z[x_1(n)] + \beta Z[x_2(n)]$$

Eg :-

1, Find the ZT of $x(n) = a^n u(n) + b^n u(-n-1)$

i, $b > a$ ii, $b < a$

$$Z[x(n)] = Z[a^n u(n)] + Z[b^n u(-n-1)].$$

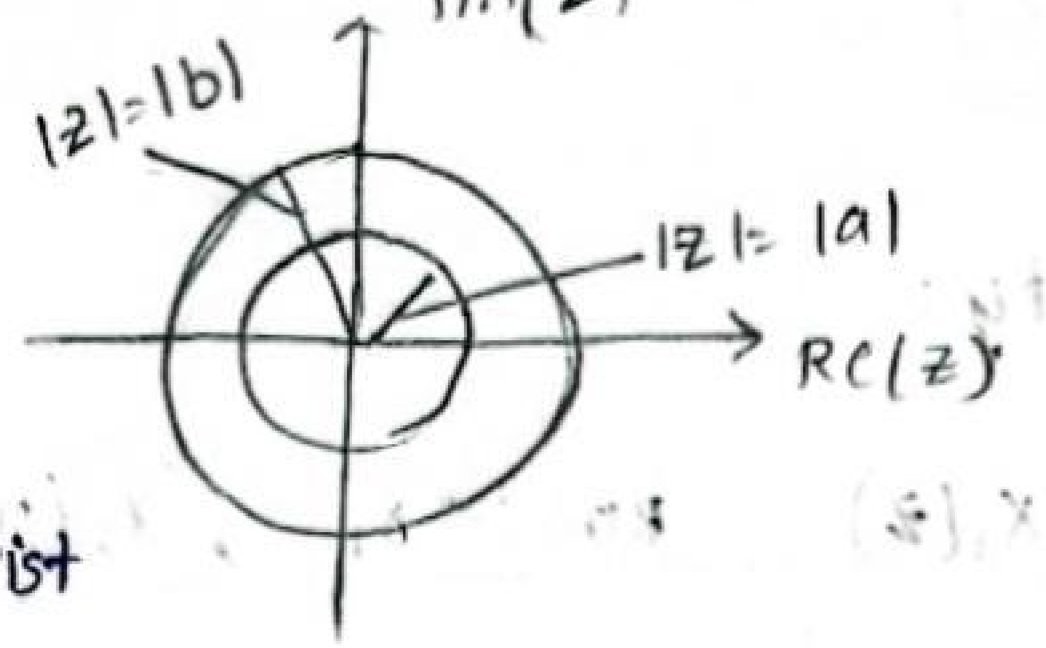
$$X(z) = \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}}$$

anticausal

ROC - $|z| > |a|$ ROC - $|z| < |b|$

ii, $b < a$

There is no common ROC.

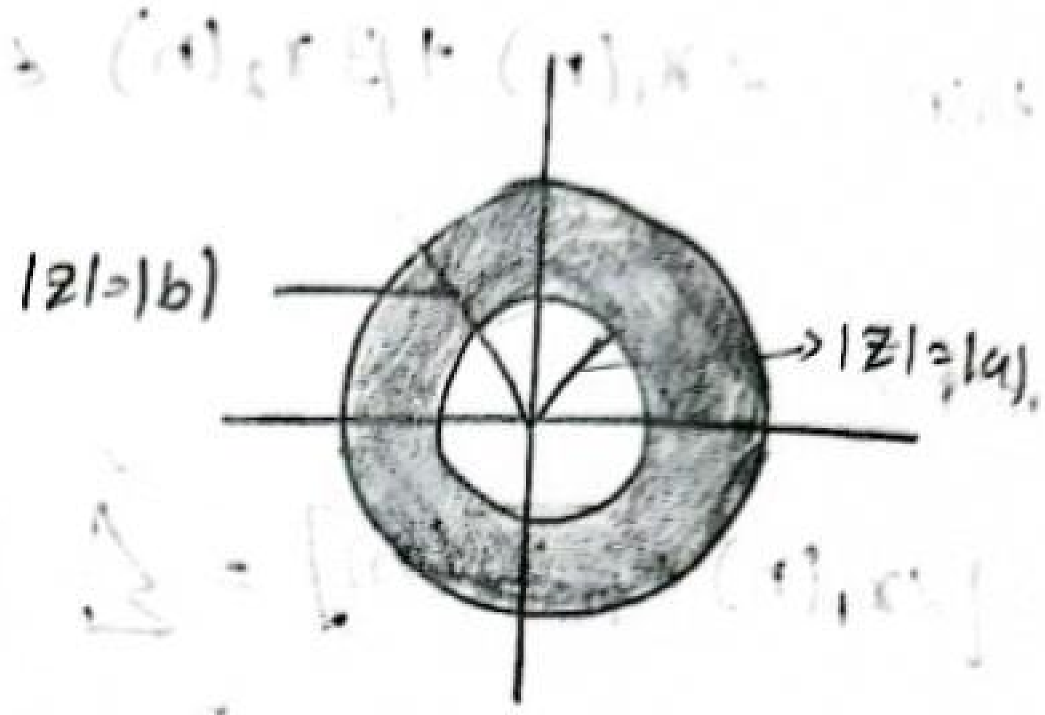


$\therefore$ ZT doesn't exist

i, $b > a$

ROC:- Annular region

$|a| < z < |b|$

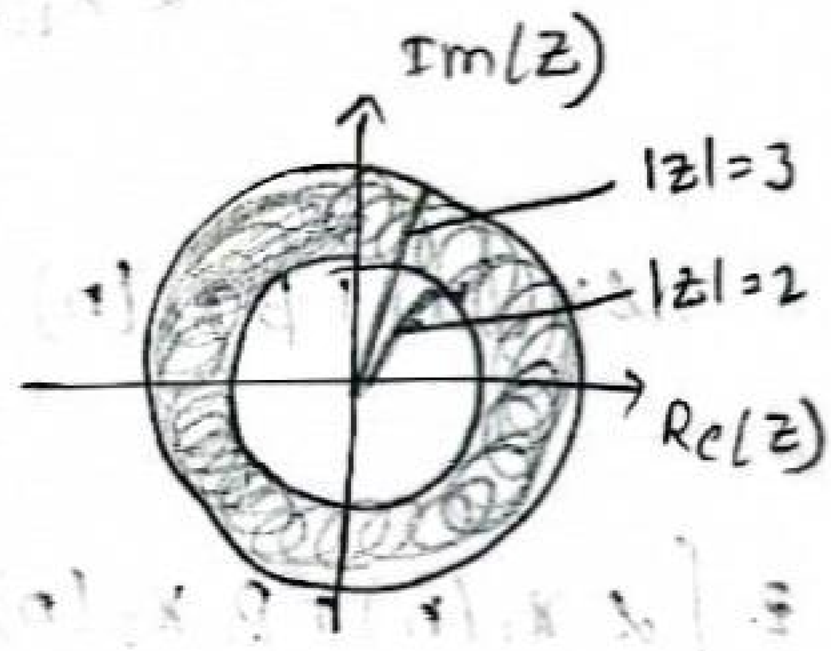


Q, Find ZT of $x(n) = 3(2)^n u(n) - 4(3)^n u(-n-1)$

$$Z[x(n)] = Z[3(2^n)u(n)] - Z[4(3)^n u(-n-1)]$$

$$X(z) = \frac{3}{1-2z^{-1}} + \frac{4}{1-3z^{-1}}$$

$|z| > 2$ $\quad$ $|z| < 3$



Q, Find the Z.T of the following signals :-

i, $x(n) = \cos(\omega_0 n)u(n)$

$$x(n) = \left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right) u(n)$$

$$= \frac{1}{2}(e^{j\omega_0})^n u(n) + \frac{1}{2}(e^{-j\omega_0})^n u(n)$$

$$Z[x(n)] = \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}}\right] + \frac{1}{2}\left[\frac{1}{1-e^{-j\omega_0}z^{-1}}\right]$$

$|z| > 1 \Leftarrow |z| > |e^{j\omega_0}|$ $\qquad$ $|z| > |e^{-j\omega_0}| \Rightarrow |z| > 1$

$$X(z) = \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{1-e^{-j\omega_0}z^{-1}}\right] \qquad ROC :- |z|>1$$

$$= \frac{1}{2} \frac{1-e^{-j\omega_0}z^{-1} + 1-e^{j\omega_0}z^{-1}}{1-e^{-j\omega_0}z^{-1} - e^{j\omega_0}z^{-1} + e^{0}(z^{-1})^2}$$

$$= \frac{1}{2} \frac{\cancel{2} - z^{-1}\left((e^{j\omega_0} + e^{-j\omega_0}\right)}{1+z^{-2} - z^{-1}(e^{j\omega_0} + e^{-j\omega_0})} = \frac{\cancel{2}}{\cancel{2}}\left[\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}}\right]$$

$$\boxed{X(z) = \frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}} \qquad \boxed{ROC :- |z|>1}$$

ii, $x(n) = \sin(\omega_0 n)u(n)$

$$x(n) = \left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2}\right) u(n)$$

$$= \frac{1}{2}(e^{j\omega_0})^n u(n) - \frac{1}{2}(e^{-j\omega_0})^n u(n)$$

$$z[x(n)] = \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}}\right] - \frac{1}{2}\left[\frac{1}{1-e^{-j\omega_0}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} - \frac{1}{1-e^{-j\omega_0}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{1} - e^{-j\omega_0}z^{-1} - \cancel{1} + e^{+j\omega_0}z^{-1}}{1-e^{-j\omega_0}z^{-1} - e^{j\omega_0}z + e^{0}z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{z^{-1}\left[e^{j\omega_0} - e^{-j\omega_0}\right]}{1+z^{-2} - z^{-1}\left[e^{j\omega_0} + e^{-j\omega_0}\right]}\right] = \frac{1}{\cancel{2}}\left[\frac{2z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos\omega_0 - z^{-2}}\right]$$

$$\boxed{X(z) = \frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}} \qquad \boxed{ROC :- |z|>1}$$

2) Time Reversal Property :-

If $x(n) \xleftrightarrow{ZT} X(z)$ ROC :- $\gamma_1 < |z| < \gamma_2$ then

$$x(-n) \xleftrightarrow{ZT} X(z^{-1}) \quad \text{ROC :- } \gamma_2 < |z| < \gamma_1$$

Proof :-

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Put $-n = m$

$$= \sum_{m=\infty}^{-\infty} x(m) z^{m}$$

summation from ∞ to $-\infty$

$$= \sum_{m=-\infty}^{\infty} x(m) (z^{-1})^{-m}$$

$= -\infty$ to ∞

$$= X(z^{-1}).$$

$$\boxed{x(-n) \xleftrightarrow{ZT} X(z^{-1})}$$

Q) Find ZT of $x(n) = u(-n)$.

WKT $y(n) = u(n) \xleftrightarrow{ZT} Y(z) = \dfrac{1}{1-z^{-1}}$ ROC :- $|z| > 1$

from time Reversal

$$y(-n) = u(-n) \xleftrightarrow{ZT} Y(z^{-1}) = \frac{1}{1-z}$$

$\boxed{\text{ROC :- } |z| < 1}$

$$\boxed{u(-n) \xleftrightarrow{ZT} \frac{1}{1-z}}$$

3, Time shifting property

If: $x(n) \xleftrightarrow{ZT} X(z)$: then $x(n-m) \xleftrightarrow{ZT} X(z)\, z^{-m}$

Proof :-

$$Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n}$$

Put $n-m = K \Rightarrow n = K+m$

$$= \sum_{K=-\infty}^{\infty} x(K)\, z^{-(K+m)}$$

$$= \sum_{K=-\infty}^{\infty} x(K)\, z^{-K}\, z^{-m} \Rightarrow z^{-m} \sum_{K=-\infty}^{\infty} x(K)\, z^{-K}$$

$$= X(z)\, z^{-m}$$

$$= X(Z)\, z^{-m}$$

$$Z[x(n-m)] = X(Z)\, z^{-m}$$

$$\boxed{x(n-m) \xleftrightarrow{ZT} X(Z)\, z^{-m}}$$

Illy $\boxed{x(n+m) \xleftrightarrow{ZT} X(Z)\, z^{m}}$

Q, Find ZT of $x(n) = \delta(n-1)$ -

WKT $y(n) = \delta(n) \xleftrightarrow{ZT} Y(z) = 1$

T.S.P

$y(n-1) = \delta(n-1) \xleftrightarrow{ZT} Y(z)\, z^{-1}$

$$\boxed{\delta(n-1) \xleftrightarrow{ZT} z^{-1}}$$

Q, $x(n) = u(n-1)$

$$y(n) = u(n) \xleftrightarrow{ZT} Y(z) = \frac{1}{1-z^{-1}} \qquad |z|>1$$

$$y(n-1) \xleftrightarrow{ZT} Y(z)\, z^{-1}$$

$$\boxed{u(n-1) \xleftrightarrow{ZT} \frac{1}{1-z^{-1}} \cdot z^{-1}} \qquad |z|>1$$

Q, $x(n) = u(n) - u(n-3)$

$$X(z) = \frac{1}{1-z^{-1}} - \frac{z^{-3}}{1-z^{-1}}$$

$$= \frac{1-z^{-3}}{1-z^{-1}} \qquad |z|>1$$

4) scaling in z-domain property :-

If $x(n) \xleftrightarrow{ZT} X(z)$ ROC:- $\gamma_1 < |z| < \gamma_2$ then

$a^n x(n) \xleftrightarrow{ZT} X\left(\frac{z}{a}\right)$, ROC :- $|a|\gamma_1 < |z| < |a| \cdot \gamma_2$

Proof:-

$$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} [a^n x(n)] \cdot z^{-n}$$

$$= \sum_{n=-\infty} x(n) \left(\frac{1}{a^{-n}}\right) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$$

$$\boxed{a^n x(n) \xleftrightarrow{ZT} X\left(\frac{z}{a}\right)}$$

$\Rightarrow$ Put $a = -1$

$$\boxed{(-1)^n x(n) \xleftrightarrow{ZT} X(-z)}$$

Q, $x(n) = a^n \cos(\omega_0 n) u(n)$

$$y(n) = \cos(\omega_0 n) u(n) \xleftrightarrow{ZT} Y(z) = \frac{1 - z^{-1}\cos(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}}$$

ROC: $|z| > 1$

apply time scaling

$$a^n y(n) \xleftrightarrow{ZT} Y\left(\frac{z}{a}\right)$$

$$\boxed{a^n \cos(\omega_0 n) u(n) \xleftrightarrow{ZT} \frac{1 - a(z)^{-1}\cos(\omega_0)}{1 - 2a(z)^{-1}\cos(\omega_0) + a^2 z^{-2}}}$$

$|z| > |a|$

Ily, $$\boxed{a^n \sin(\omega_0 n) u(n) \longleftrightarrow \frac{a z^{-1}\sin(\omega_0)}{1 - 2az^{-1}\cos(\omega_0) + a^2 z^{-2}}}$$

$|z| > |a|$

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5, complex conjugation property :-

If $x(n) \xleftrightarrow{ZT} X(z)$ ROC: R then $x^*(n) \xleftrightarrow{ZT} X^*(z^*)$

ROC : R.

Proof :-

$$Z[x^*(n)] = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x^*(n) \left[(z^*)^*\right]^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x^*(n) \left[(z^*)^n \right]^*$$

$$= \sum_{n=-\infty}^{\infty} \left[x(n) (z^*)^{-n} \right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n} \right]^*$$

$$= X^*[z^*]$$

$$\boxed{x^*(n) \longleftrightarrow X^*(z^*)}$$

Q, find Z.T of signal $x^*(-n)$

$$y(n) = x(-n) \xleftrightarrow{ZT} Y(z) = X(z^{-1})$$

$$y^*(n) \longleftrightarrow Y^*(z^*)$$

$$\boxed{x^*(-n) \xleftrightarrow{ZT} X^*\left(\frac{1}{z^*}\right)}$$

→ If the signal is a real signal $x(n) = x^*(n)$

Apply Z.T on B.S

$$z[x(n)] = z[x^*(n)]$$

$$\boxed{X(z) = X^*(z^*)}$$

If even & real $X(z)$, $X(z^+)$

even signal $x(n) = x(-n)$

$X(z) = X(z^{-1}) = 0$

If $z = z_0$ is zero of $X(z)$, then $X(z_0) = X^*(z_0^*) = 0$

$P = P_0$ pole " " then $X(P_0) = X^*(P_0^*) = \infty$

even signal $\Rightarrow X(P_0) = X\left(\frac{1}{P_0}\right) = \infty$

6) **Difference property** :- → Analogous to diffe

If $x(n) \xleftrightarrow{ZT} X(z)$ ROC: R then :

$$x(n) - x(n-1) \xleftrightarrow{ZT} (1-z^{-1})X(z) \qquad \text{ROC}: R - \{0\}$$

Proof:-

$$Z[x(n) - x(n-1)] = Z[x(n)] - Z[x(n-1)]$$

$$= X(z) - z^{-1}X(z)$$

$$= (1-z^{-1})X(z)$$

$$\boxed{Z[x(n) - x(n-1)] \xleftrightarrow{ZT} (1-z^{-1})X(z)}$$

7) **Differentiation in z-domain property** :-

If $x(n) \xleftrightarrow{ZT} X(z)$ ROC : R then $n\,x(n) \xleftrightarrow{ZT} z\frac{dX(z)}{dz}$

ROC: $R - \{\infty\}$

Proof:-

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{d}{dz} z^{-n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-n)z^{-n-1}$$

$$= z^{-1} \sum_{n=-\infty}^{\infty} -n\, x(n) z^{-n}$$

$$z\frac{dX(z)}{dz} = \underbrace{\sum_{n=-\infty}^{\infty} -n\,x(n)\,z^{-n}}_{ZT[x(n)]}$$

$$z\frac{dX(z)}{dz} \xleftrightarrow{ZT} -n\,x(n)$$

$$\boxed{-n\,x(n) \xleftrightarrow{ZT} z\frac{dX(z)}{dz}}$$

Q, Find the Z.T $x(n) = n a^n u(n)$

$$y(n) = a^n u(n) \xleftrightarrow{ZT} Y(z) = \frac{1}{1-az^{-1}}$$

Apply differentiation in z-domain

$$n\,y(n) \xleftrightarrow{ZT} -z\frac{dY(z)}{dz}$$

$$n a^n u(n) \xleftrightarrow{ZT} -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right)$$

$$\boxed{n a^n u(n) \xleftrightarrow{ZT} \frac{az^{-1}}{(1-az^{-1})^2}}$$ $\boxed{ROC: |z| > |a|}$

Q, Find Z.T of ramp signal $[n\,u(n)]$

$$n a^n u(n) \xleftrightarrow{ZT} \frac{az^{-1}}{(1-az^{-1})^2}$$

Put $a = 1$

$$\boxed{n\,u(n) \xleftrightarrow{ZT} \frac{z^{-1}}{(1-z^{-1})^2}}$$

Q, Find ZT of $x(n) = n^2 a^n u(n)$

$$y(n) = n a^n u(n) \longleftrightarrow Y(z) = \frac{a z^{-1}}{(1 - a z^{-1})^2}$$

Apply diff in z domain

$$n\, y(n) \xleftrightarrow{ZT} -z \frac{d\, Y(z)}{dz}$$

$$n^2 a^n u(n) \longleftrightarrow -z \frac{d}{dz}\left[\frac{a z^{-1}}{(1 - a z^{-1})^2}\right]$$

$$\boxed{n^2 a^n u(n) \xleftrightarrow{ZT} \frac{a z^{-1}(1 + a z^{-1})}{(1 - a z^{-1})^3}}$$

8, convolution theorem :-

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$ ROC: r_1, $x_2(n) \xleftrightarrow{ZT} X_2(z)$ ROC: r_2

then $x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z) \cdot X_2(z)$ ROC: $r_1 \cap r_2$

Proof :-

$$x_1(n) * x_2(n) = \sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(n-K)$$

$$z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(n-K)\right] z^{-n}$$

$$= \sum_{K=-\infty}^{\infty} x_1(K) \underbrace{\sum_{n=-\infty}^{\infty} x_2(n-K)\, z^{-n}}_{z[x_2(n-K)] = z^{-K} X_2(z)}$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k}}_{X_1(z)} X_2(z)$$

$$= X_1(z) \cdot X_2(z)$$

$$\boxed{x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z) \cdot X_2(z)}$$

- $Z[x_1(n) * x_2(n)] = Z[x_1(n)] \cdot Z[x_2(n)]$

Q, Find convolution b/w the 2 following signals

$x_1(n) = \{1, -2, 1\}$

$$x_2(n) = \begin{cases} 1 & \text{for } 0 \le n \le 5 \\ 0 & \text{else} \end{cases}$$

$$X_1(z) = \sum_{n=0}^{2} x_1(n) z^{-n}$$

$$= x_1(0) z^0 + x_1(1) z^{-1} + x_1(2) z^{-2}$$

$$= 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = \sum_{n=0}^{5} x_2(n) z^{-n}$$

$$= x_2(0) + x_2(1) z^{-1} + x_2(2) z^{-2} + x_2(3) z^{-3} + x_2(4) z^{-4} + x_2(5) z^{-5}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$Y(z)$

$$= X_1(z) \cdot X_2(z) = 1 + \cancel{z^{-1}} + \cancel{z^{-2}} + \cancel{z^{-3}} + \cancel{z^{-4}} + \cancel{z^{-5}} - 2z^{-1} - \cancel{2z^{-2}} - \cancel{2z^{-3}} - \cancel{2z^{-4}}$$
$$- \cancel{2z^{-5}} - 2z^{-6} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7}$$

$= 1 - z^{-1} - z^{-6} + z^{-7}$

$y(n) = x_1(n) * x_2(n) = IZT[X_1(z) \cdot X_2(z)]$

$= IZT[Y(z)]$

$$Y(z) = \sum_{n=0}^{7} y(n) z^{-n}$$

$Y(z) = y(0) + y(1)z^{-1} + \cdots\cdots + y(7)z^{-7}$

$y(n) = \{1, -1, 0, 0, 0, 0, -1, 1\}$

9, Correlation theorem :-

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$ ROC : r_1 , $x_2(n) \xleftrightarrow{ZT} X_2(z)$ ROC: r_2

then $\gamma_{x_1 x_2}(m) \xleftrightarrow{ZT} X_1(z) \cdot X_2(z^{-1})$

- Cross correlation function b/w $x_1(n)$ & $x_2(n)$ is defn'd as

$$\boxed{\gamma_{x_1,x_2}(m) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2(n-m)}$$

Proof:-

$$x_1(m) * x_2(-m) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2(n-m)$$

$\gamma_{x_1,x_2}(m) = x_1(m) * x_2(-m)$

$Z[\gamma_{x_1,x_2}(m)] = Z[x_1(m) * x_2(-m)]$

$= Z[x_1(m)] \cdot Z[x_2(-m)]$

$$\boxed{Z[\gamma_{x_1,x_2}(m)] = X_1(z) \cdot X_2(z^{-1})}$$

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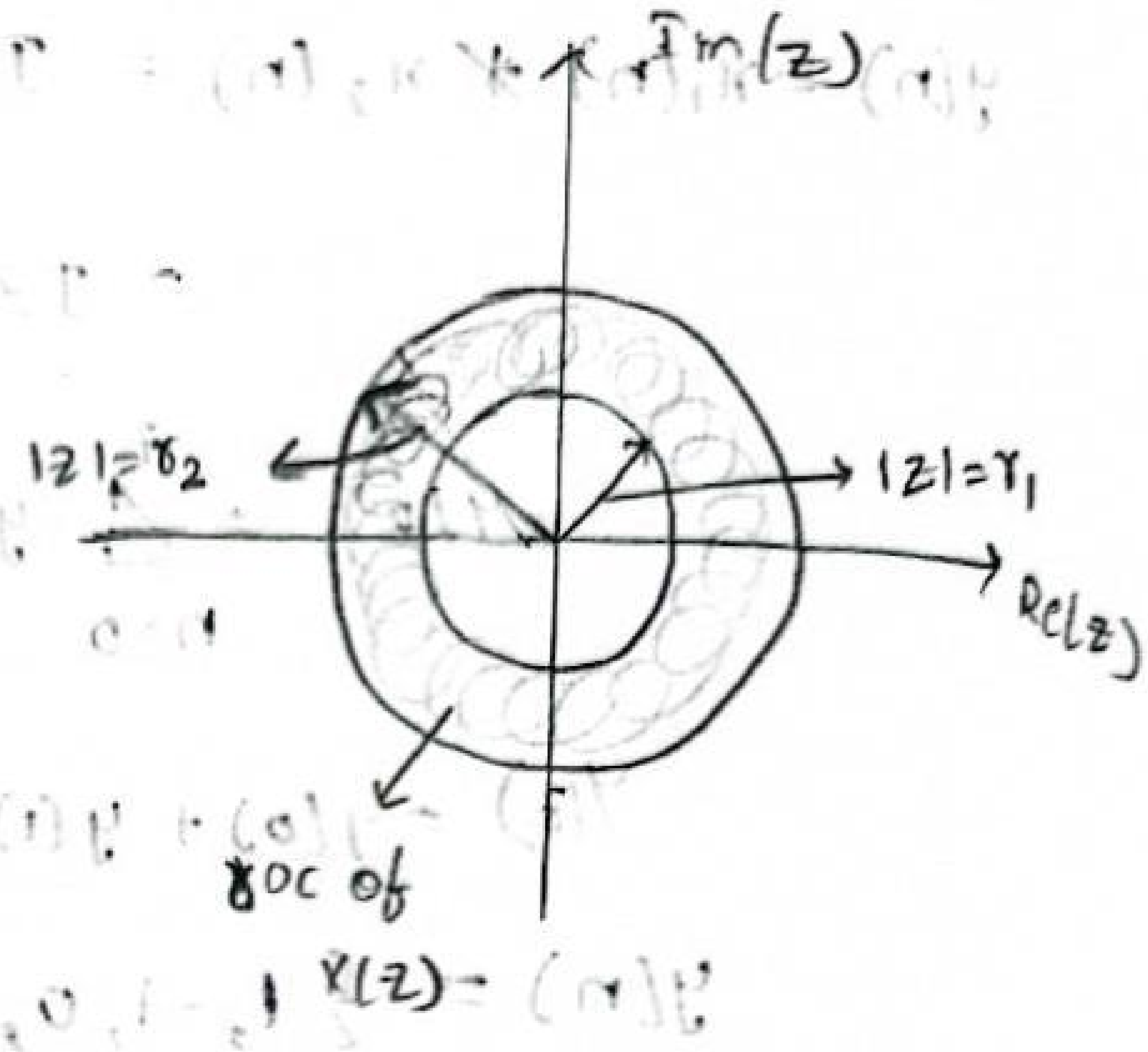
$r_1 < |z| < r_2$

10) Inverse z-Transform :-



$$x(n) \xleftrightarrow{ZT} X(z)$$

$$X(z) = \sum_{K=-\infty}^{\infty} x(K) z^{-K}$$

$$X(z)\, z^{n-1} = \sum_{n=-\infty}^{\infty} x(K)\, z^{n-1-K}$$

$$\oint_{c} X(z) \cdot z^{n-1} dz = \sum_{K=-\infty}^{\infty} x(n) \oint_{c} z^{n-1-k}\, dz$$

Cauchy integral theorem:- $\frac{1}{2\pi j} \oint_{c} z^{n-1-k}\, dz = \begin{cases} 1 & \text{for } n=k \\ 0, & \text{else} \end{cases}$

$$\oint_{c} X(z)\, z^{n-1} dz = [0 + 0 + x(n) 2\pi j + 0 + 0 \cdots]$$

$$= 2\pi j\, x(n)$$

IZT $\boxed{x(n) = \frac{1}{2\pi j} \oint_{c} X(z)\, z^{n-1} dz}$

11) multiplication theorem:-

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$ ROC :- $r_{1l} < |z| < r_{1u}$

$x_2(n) \xleftrightarrow{ZT} X_2(z)$ ROC :- $r_{2l} < |z| < r_{2u}$

then $x_1(n) \cdot x_2(n) \xleftrightarrow{ZT} \frac{1}{2\pi j} \oint_{c} X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1} dv$

ROC :- $r_{1l}\, r_{2l} < |z| < r_{1u}\, r_{2u}$

Proof:-

$$Z[x_1(n).x_2(n)] = \sum_{n=-\infty}^{\infty} x_1(n).x_2(n).z^{-n}$$

$$x_1(n) = \frac{1}{2\pi j}\oint_c X_1(v).v^{n-1}dv$$ where v is dummy variable

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi j}\oint_c X_1(v).v^{n-1}dv\right]x_2(n).z^{-n}$$

$$= \frac{1}{2\pi j}\oint_c X_1(v)\left[\sum_{n=-\infty}^{\infty} x_2(n).v^{n-1}.z^{-n}\right]dv$$

$$= \frac{1}{2\pi j}\oint_c X_1(v)\underbrace{\left[\sum_{n=-\infty}^{\infty} x_2(n).\left(\frac{z}{v}\right)^{-n}\right]}_{X_2\left(\frac{z}{v}\right)}v^{-1}dv$$

$$\boxed{Z[x_1(n).x_2(n)] = \frac{1}{2\pi j}\oint_c X_1(v).X_2\left(\frac{z}{v}\right).v^{-1}dv}$$

12> Parsevals Theorem:-

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_c X_1(v).X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv$$

$$\Rightarrow Z[x_1(n)x_2^*(n)] = \sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n).z^{-n}$$

$$= \frac{1}{2\pi j}\oint_c X_1(v).X_2^*\left(\frac{z^*}{v^*}\right)v^{-1}dv$$

put $z=1$

$$\boxed{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv}$$

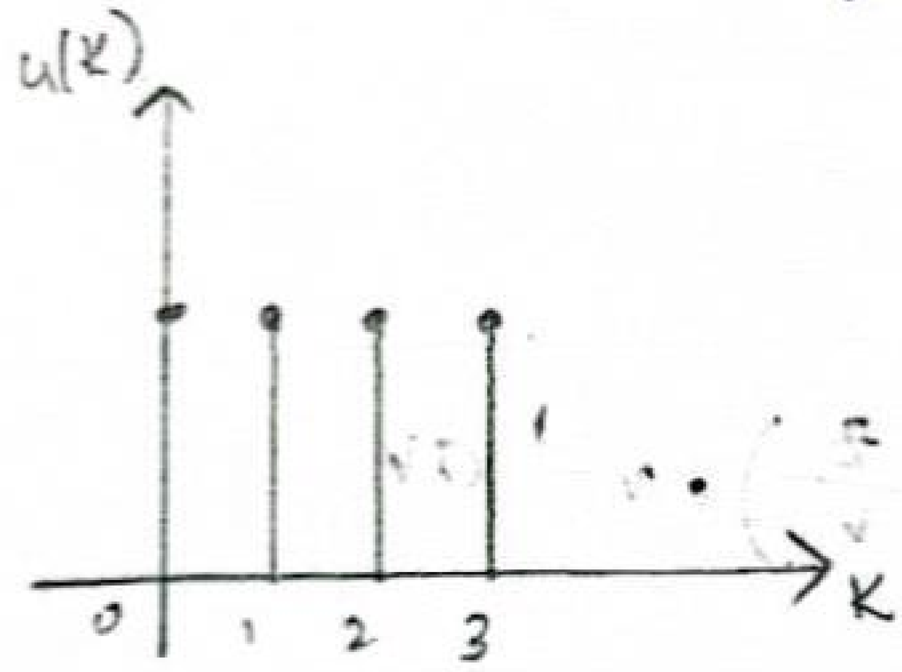
13) Accumulation property

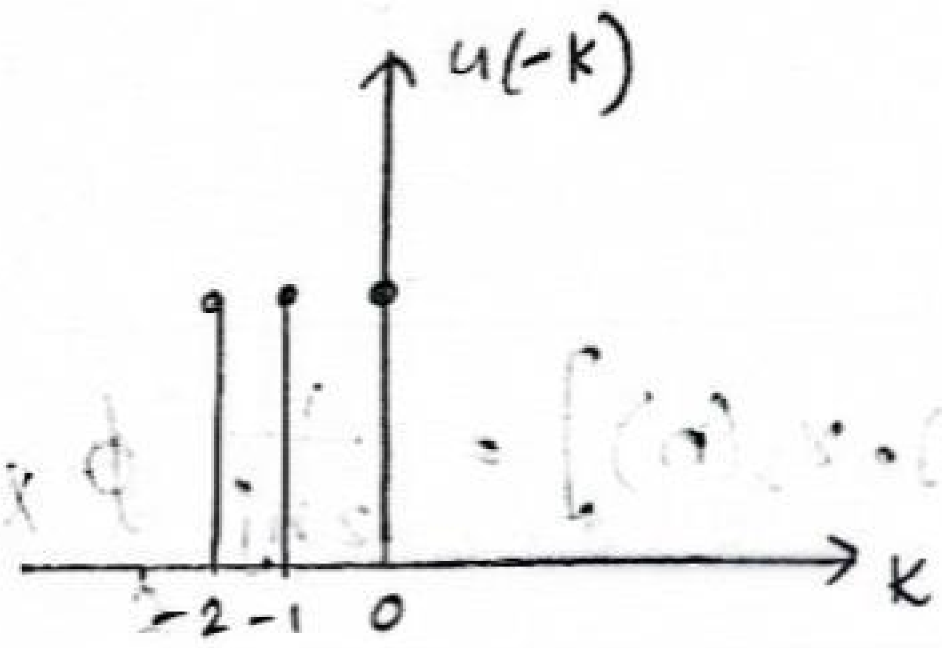
If $x(n) \xleftrightarrow{ZT} X(z)$ ROC = R then

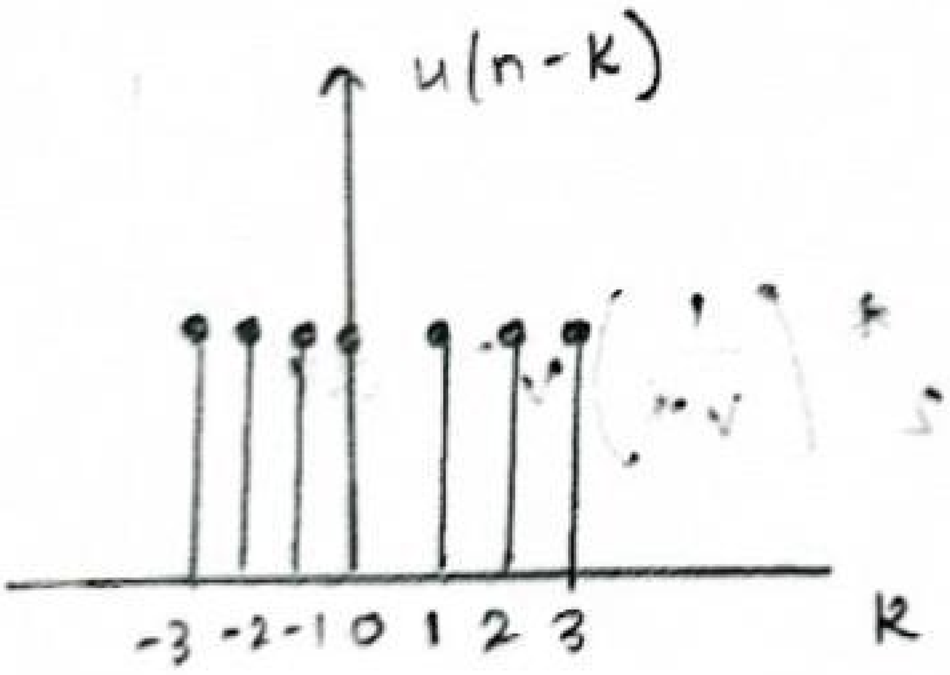
$$\sum_{K=-\infty}^{n} x(K) \xleftrightarrow{ZT} \frac{X(z)}{1-z^{-1}} \qquad \text{ROC}: R - \{1\}$$

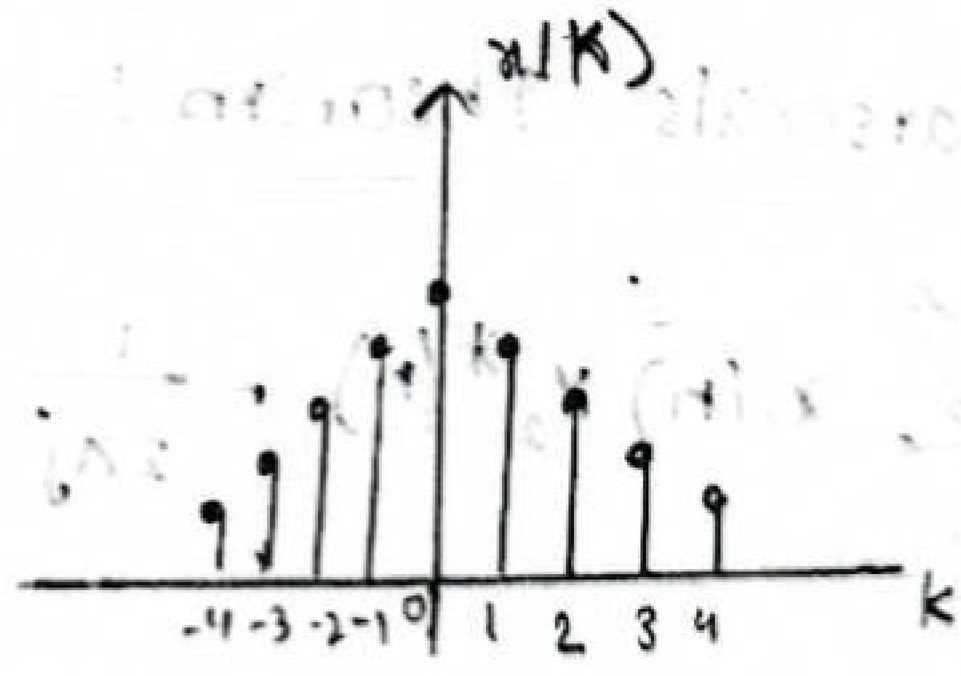
Proof :-

$$x(n) * u(n) = \sum_{K=-\infty}^{\infty} x(K)\, u(n-K)$$









$$x(n) * u(n) = \sum_{K=-\infty}^{n} x(K)(1)$$

$$\sum_{K=-\infty}^{n} x(K) = x(n) * u(n)$$

$$Z\left[\sum_{K=-\infty}^{n} x(K)\right] = Z\left[x(n) * u(n)\right]$$

$$= Z\left[x(n)\right] \cdot Z\left[u(n)\right]$$

$$= X(z) \cdot \frac{1}{1-z^{-1}} = \frac{X(z)}{1-z^{-1}}$$

$$\boxed{\sum_{K=-\infty}^{n} x(K) \xleftrightarrow{ZT} \frac{X(z)}{1-z^{-1}}}$$

Methods of IZT calculation

1. IZT using partial fraction method

IMP

Q. Find the IZT of the following

i. $X(z) = \dfrac{1}{1-3z^{-1}+2z^{-2}}$

(a) $|z|<1$ (b) $|z|>2$ (c) $1<|z|<2$

$$X(z) = \frac{1}{z^{-2}(z^2-3z+2)} = \frac{z^2}{z^2-3z+2}$$

$$\frac{X(z)}{z} = \frac{z}{(z^2-3z+2)} = \frac{z}{(z-1)(z-2)} = \frac{C_1}{z-1} + \frac{C_2}{z-2}$$

$z=1 \Rightarrow 1 = -C_1 \Rightarrow C_1 = -1$

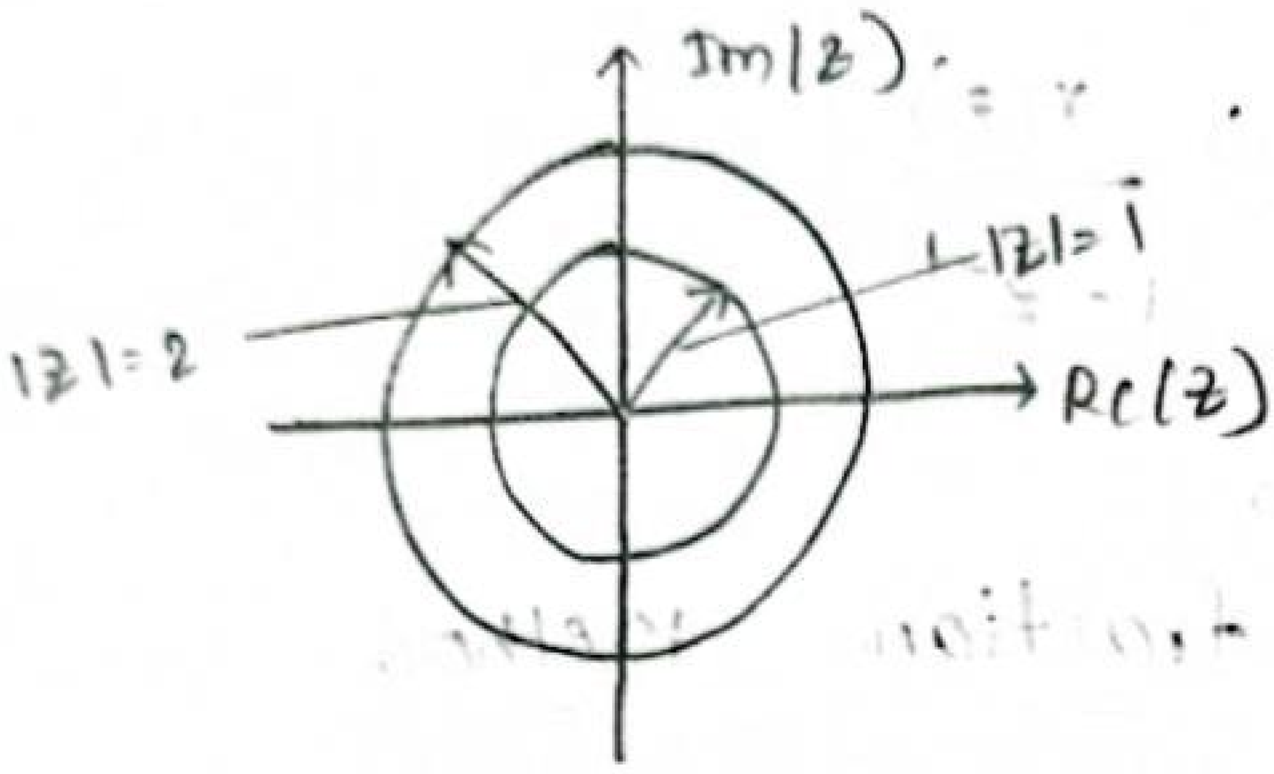
$z=2 \Rightarrow 2 = 1C_2 \Rightarrow C_2 = 2$

$$\frac{X(z)}{z} = \frac{-1}{z-1} + \frac{2}{z-2}$$

$$= \frac{-z}{z-1} + \frac{2z}{z-2}$$

$$= \frac{-1}{1-z^{-1}} + \frac{2}{1-2z^{-1}}$$

Poles $z=1, z=2$



a) $|z|<1$ $\Rightarrow$ the signal is anti causal

$$x(n) = u[-n-1] - 2\cdot 2^n u[-n-1]$$

$\frac{1}{1-az^{-1}}$ causal $a^n \cdot u(n)$; anticausal $-a^n \cdot u(-n-1)$

b) $|z|>2$ $\Rightarrow$ the signal must be a causal signal

$$x(n) = -u(n) + 2(2)^n \cdot u(n)$$

c) $1<|z|<2$ $\Rightarrow$ the signal is a 2-sided signal

$|z|>1$ & $|z|<2$

$|z|>1$ $\downarrow$ the pole corresponding to $z=1$ is causal

$|z|<2$ $\hookrightarrow$ the pole corresponding to $z=2$ is anticausal

$x(n) = -u(n) - 2 \cdot 2^n u(-n-1)$

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(ii) $X(z) = \dfrac{1}{(1+z^{-1})(1-z^{-1})^2}$ ROC : $|z| > 1$

$= \dfrac{1}{z^{-1}(z+1) \cdot z^{-2}(z-1)^2} = \dfrac{z^3}{(z+1)(z-1)^2}$

$\dfrac{X(z)}{z} = \dfrac{z^2}{(z+1)(z-1)^2} = \dfrac{c_1}{z+1} + \dfrac{\lambda_1}{z-1} + \dfrac{\lambda_2}{(z-1)^2}$

method

(1) If $z = -1 \Rightarrow 1 = c_1(4) \Rightarrow c_1 = 1/4$

$z^2 = c_1(z-1)^2 + \lambda_1(z-1)(z+1) + \lambda_2(z+1)$

$z^2 = z^2(c_1 + \lambda_1) + z(-2c_1 + \lambda_2) + c_1 - \lambda_1 + \lambda_2$

$1 = c_1 + \lambda_1 \Rightarrow 1 - \frac{1}{4} = \lambda_1 \Rightarrow \lambda_1 = 3/4$

$0 = -2c_1 + \lambda_2 \Rightarrow \lambda_2 = 2 \frac{1}{4} = 1/2$

(2) $c_1 = (z+1) \cdot \dfrac{X(z)}{z}\Big|_{z=-1} = 1/4$

$\lambda_2 = (z-1)^2 \cdot \dfrac{X(z)}{z}\Big|_{z=1} = 1/2$

$\lambda_1 = \dfrac{d}{dz}\left[(z-1)^2 \dfrac{X(z)}{z}\right]\Big|_{z=1} = 3/4$

$$X(z) = \frac{1}{4}\left(\frac{z}{z+1}\right) + \frac{3}{4}\left(\frac{z}{z-1}\right) + \frac{1}{2}\left(\frac{z}{(z-1)^2}\right)$$

$$X(z) = \frac{1}{4}\left(\frac{1}{1+z^{-1}}\right) + \frac{3}{4}\left(\frac{1}{1-z^{-1}}\right) + \frac{1}{2}\left(\frac{z^{-1}}{(1-z^{-1})^2}\right)$$

$$= \frac{1}{4}(-1)^n u(n) + \frac{3}{4} u(n) + \frac{1}{2} n u(n)$$

iii, $X(z) = \dfrac{z^3+z^2}{(z-1)(z-3)}$

$$\frac{X(z)}{z} = \frac{z^2+z}{(z-1)(z-3)} = \frac{z^2+z}{z^2-4z+3}$$

$$z^2-4z+3 \overline{\big)\, z^2+z \,}(1$$
$$z^2-4z+3$$
$$5z-3$$

$$\frac{X(z)}{z} = 1 + \frac{5z-3}{(z-1)(z-3)}$$

$$\frac{5z-3}{(z-1)(z-3)} = \frac{C_1}{z-1} + \frac{C_2}{z-3}$$

If $z=1 \Rightarrow 2 = -2C_1 \Rightarrow C_1 = -1$

$z=3 \Rightarrow 12 = 2C_2 \Rightarrow C_2 = 6$

$$\frac{X(z)}{z} = 1 + \frac{-1}{(z-1)} + \frac{6}{(z-3)}$$

$$X(z) = z - \frac{z}{z-1} + \frac{6z}{z-3}$$

$$= z - \frac{1}{1-z^{-1}} + \frac{6}{1-3z^{-1}}$$ $z \leftrightarrow \delta(n+1)$

$$x(n) = \delta(n+1) - u(n) + 6(3)^n u(n).$$

2, IZT using Residue Method

$$x(n) = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$$

= sum of residues of $X(z)z^{n-1}$, at the poles inside 'c'

z_i = poles of $X(z)$

- residue of $X(z) z^{n-1}$ is calculated $= \sum_{i=1}^{N} (z-z_i) X(z) z^{n-1} \Big|_{z=z_i}$

- If $X(z)$ has repeated pole of order k, the residue at that pole is given by

$$\frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z-z_i) X(z) z^{n-1} \right] \Big|_{z=z_i}$$

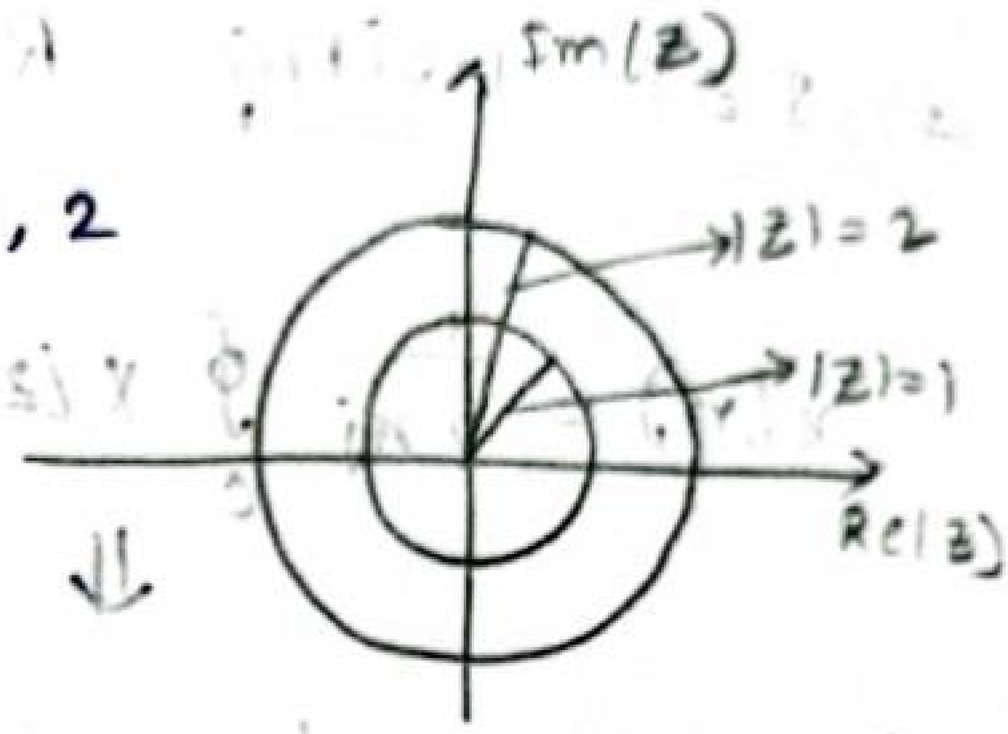
Q, Find the IZT of following using residue method

i, $X(z) = \dfrac{z(z+1)}{z^2-3z+2}$

a, $|z|>2$ b, $|z|<1$ c, $1<|z|<2$

$$X(z) = \frac{z(z+1)}{(z-1)(z-2)}$$

Poles = 1, 2



$$R_1 : (z-z_1)\, X(z)\, z^{n-1} \Big|_{z=z_1}$$

$$= \cancel{(z-1)} \left[\frac{z(z+1)}{\cancel{(z-1)}(z-2)} \right] z^{n-1} \Big|_{z=1}$$

$$= \frac{1(1+1)}{(1-2)} (1)^{n-1}$$

$$= -2\,(1)^{n-1} = -2\,(1)^n$$

$$R_2 : \cancel{(z-2)} \left[\frac{z(z+1)}{(z-1)\cancel{(z-2)}} \right] z^{n-1} \Big|_{z=2}$$

$$= \frac{2(3)}{(1)}\, 2^{n-1} = 3\,(2)^n$$

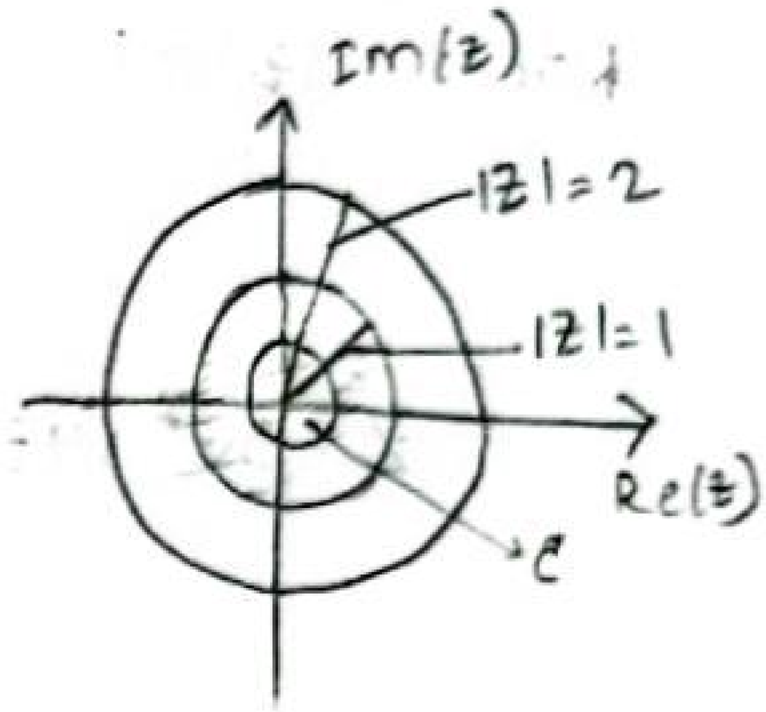
• 2 poles $z=1$, $z=2$ are inside c, Residue are +ve

a)

$$x(n) = \underbrace{(-2)(1)^n u(n)}_{\text{causal}} + 3(2)^n u(n)$$

b)

• The poles are outside of 'c' then the residue is -ve then we choose the $-R_1$, $-R_2$

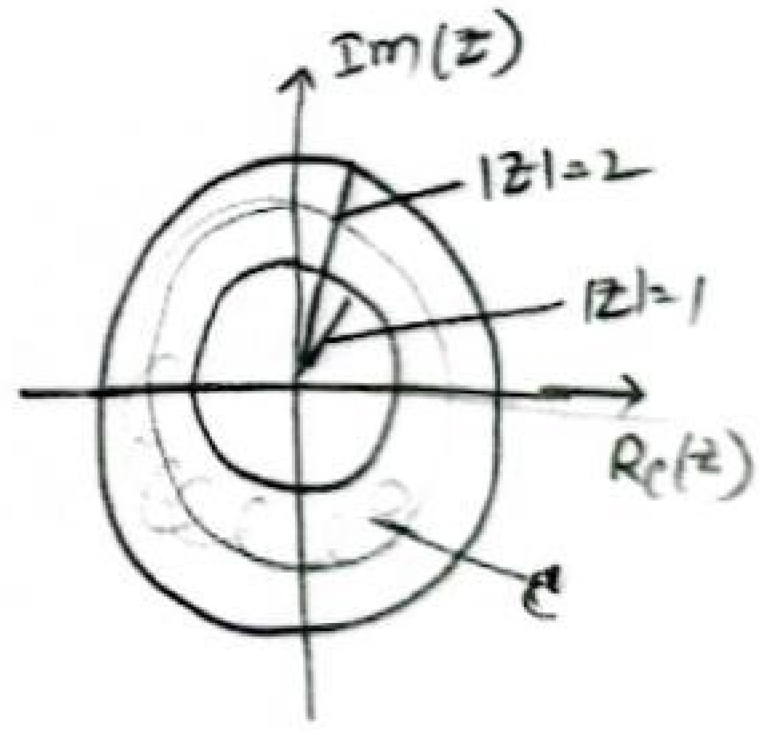


$$x(n) = 2(1)^n u(-n-1) - 3(2)^n u(-n-1).$$

c)

Here the poles are corresponding, z_1 = inside the c then it is +ve

z_2 = corresponding outside the c then it is -ve



$1 < |z| < 2$

$|z| > 1 \rightarrow$ causal

$|z| < 2 \rightarrow$ anti causal

$$x(n) = -2(1)^n \cdot u(n) - 3(2)^n u(-n-1)$$

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3, IZT using power series method /long division method:

i, $X(z) = \dfrac{z^2+2z}{1+4z-3z^2+z^3}$ ROC : $|z|>1$

ROC :- $|z|>1 \Rightarrow x(n)$ is a causal signal

$$z^3-3z^2+4z+1 \overline{\big)\ z^2+2z}\ \left(z^{-1}+5z^{-2}+11z^{-3}+12z^{-4}\right.$$

$$\begin{array}{l} \cancel{z^2}-3z+4+z^{-1} \\ \hline 5\cancel{z}-4-z^{-1} \\ 5z-15+20z^{-1}+5z^{-2} \\ \hline 11-21z^{-1}-5z^{-2} \\ 11-33z^{-1}+44z^{-2}+11z^{-3} \\ \hline 12z^{-1}-49z^{-2}-11z^{-3} \\ 12z^{-1}-36z^{-2}+48z^{-3}+12z^{-4} \\ \hline -13z^{-2}-59z^{-3}-12z^{-4} \end{array}$$

$X(z) \simeq z^{-1}+5z^{-2}+11z^{-3}+12z^{-4}$

$X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n} = x(0)+x(1)z^{-1}+x(2)z^{-2}+ \cdots$

By comparing above Eqn's

$x(n) = \{0, 1, 5, 11, 12, \cdots\}$

ii, $X(z) = \dfrac{1}{1 - z^{-3}}$ $\quad |z| > 1$

causal signal

$$X(z) = \frac{z^3}{z^3 - 1}$$

$z^3 - 1 \overline{)\, \cancel{z^3}} \quad (1 + z^{-3} + z^{-6} + z^{-9}$

$\pm \cancel{z^3} \mp 1$

$\cancel{1}$

$\cancel{1} - z^{-3}$
$- \quad +$

$\cancel{z^{-3}}$

$\cancel{z^{-3}} - z^{-6}$
$- \quad +$

$\cancel{z^{-6}}$

$\cancel{z^{-6}} - z^{-9}$
$- \quad +$

z^{-9}

$$X(z) \simeq 1 + z^{-3} + z^{-6} + z^{-9} + z^{-12} + \dots$$

$$x(n) = \{1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \dots\}$$

iii, $X(z) = \dfrac{z^2 + z + 2}{z^3 - 2z^2 + 3z + 4}$ $\quad$ ROC :- $|z| < 1$

$\downarrow$

anticausal signal

→ anti causal - ascending power of z

→ causal - descending power of z

$$4+3z-2z^2+z^3)\ \cancel{2}+z+z^2\ \left(\frac{1}{2}-\frac{1}{8}z+\frac{19}{32}z^2-\frac{81}{128}z^3\right.$$

$$\cancel{2}+\frac{3}{2}z-z^2+\frac{1}{2}z^3$$

$$-\quad-\quad+\quad-$$

$$-\frac{1}{2}\cancel{z}+2z^2-\frac{1}{2}z^3$$

$$-\frac{1}{2}z-\frac{3}{8}z^2+\frac{1}{4}z^3-\frac{1}{8}z^4$$

(−)

$$\frac{19}{8}\cancel{z^2}-\frac{3}{4}z^3+\frac{1}{8}z^4$$

$$\frac{19}{8}z^2+\frac{57}{32}z^3-\frac{19}{16}z^4+\frac{19}{32}z^5$$

(−)

$$\frac{-81}{32}\cancel{z^3}+\frac{211}{16}z^4-\frac{19}{32}z^5$$

$$-\frac{81}{32}z^3-\frac{243}{128}z^4+\frac{81}{64}z^5-\frac{81}{128}z^6$$

$$\frac{411}{128}z^4-\frac{119}{64}z^5+\frac{81}{128}z^6$$

$$x(z)=\frac{1}{2}-\frac{1}{8}z+\frac{19}{32}z^2-\frac{81}{128}z^3$$

$$x(z)=\sum_{n=-\infty}^{0}x(n)z^{-n}$$

$$x(z)=x(0)+x(1)z^{-1}+x(2)z^{2}+x(3)z^{3}+\ldots$$

$$x(n)=\left\{\ldots,\frac{-81}{128},\frac{19}{32},\frac{-1}{8},\underset{\uparrow}{\frac{1}{2}}\right\}$$

iv) IZT using convolution method:-

i) $x(z) = \dfrac{z^2}{(z-2)(z-3)}$

$x_1(n) * x_2(n) = IZT\left[X_1(z)\cdot X_2(z)\right]$

$X(z) = \left(\dfrac{z}{z-2}\right)\left(\dfrac{z}{z-3}\right)$

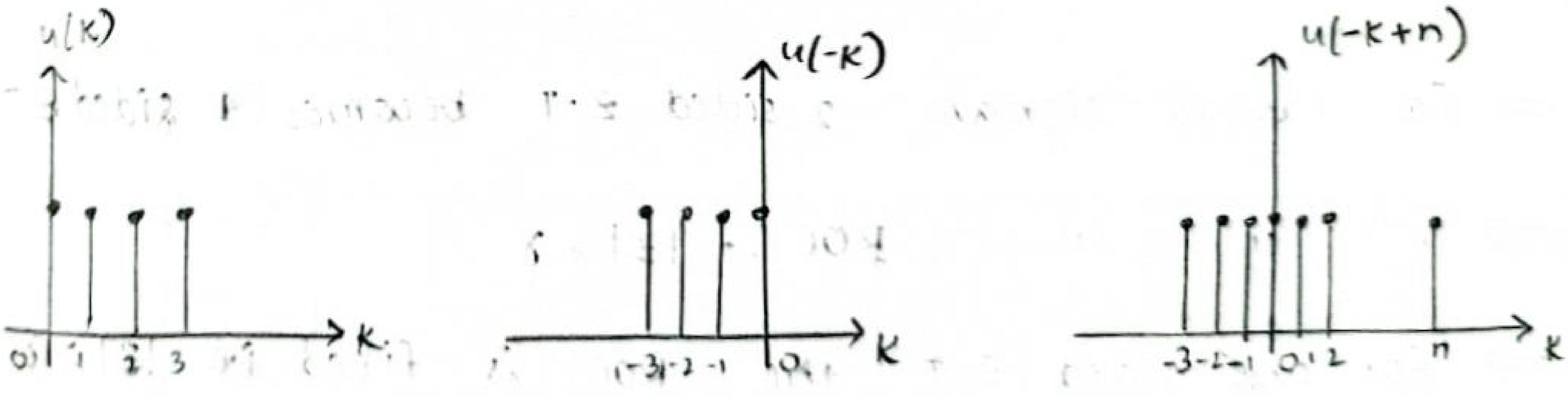
$X(z) = X_1(z)\cdot X_2(z)$

$x_1(n) = 2^n u(n) \xleftrightarrow{ZT} X_1(z) = \dfrac{z}{z-2} = \dfrac{1}{1-2z^{-1}}$

$x_2(n) = 3^n u(n) \xleftrightarrow{ZT} X_2(z) = \dfrac{1}{1-3z^{-1}}$

$$x_1(n) * x_2(n) = \sum_{K=-\infty}^{\infty} x_1(K)\,x_2(n-K)$$

$$= \sum_{K=-\infty}^{\infty} 2^K u(K)\, 3^{n-K} u(n-K)$$



$$= \sum_{K=0}^{n} 2^K 3^{n-K} = 3^n \sum_{K=0}^{n}\left(\frac{2}{3}\right)^K \qquad \left[\sum_{\gamma=0}^{n-1} \gamma^n = \frac{1-\gamma^N}{1-\gamma}\right]$$

$$= 3^n\left[\frac{1+\left(\frac{2}{3}\right)^{n+1}}{1-(2/3)}\right]$$

$$= 3^n \left[\frac{1 - \frac{2^{n+1}}{3^{n+1}}}{1/3} \right]$$

$$= 3^{n+1} \left[1 - \frac{2^{n+1}}{3^{n+1}} \right]$$

$$x(n) = (3^{n+1} - 2^{n+1}) \, u(n)$$

One sided Z·T / unilateral Z·T :-

$$x(n) \xleftrightarrow{ZT} X(z)$$

$$X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n} \rightarrow \text{1 sided Z·T}$$

- Two sided Z·T is defined as $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

If signal is causal $\Rightarrow X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$\Rightarrow$ For causal signals 2 sided Z·T becomes 1 sided Z·T

$\Rightarrow$ " " " ROC :- $|z| > r$

$\Rightarrow$ For one sided Z·T the ROC is fixed i.e $|z| > r$

Properties

1) Time delay Property :-

If $x(n) \leftrightarrow X(z)$ then $x(n+k) \xleftrightarrow{ZT} z^{-k}\left[X(z) + \sum_{l=-k}^{-1} x(k) z^{-l}\right]$

Proof :-

$$Z[x(n-k)] = \sum_{n=0}^{\infty} x(n-k) z^{-n}$$

Put $n-k = l$

$$= \sum_{l=-k}^{\infty} x(l) z^{-l} z^{-k}$$

$$= z^{-k} \left[\sum_{l=-k}^{\infty} x(l) z^{-l} \right]$$

$$= z^{-k} \left[\sum_{l=-k}^{-1} x(l) z^{-l} + \underbrace{\sum_{l=0}^{\infty} x(l) z^{-l}}_{X(z)} \right]$$

$$= z^{-k} \left[X(z) + \sum_{l=-k}^{-1} x(l) z^{-l} \right].$$

$$\boxed{x(n-k) \xleftrightarrow{ZT} z^{-k} \left[X(z) + \sum_{l=-k}^{-1} x(l) z^{-l} \right]}$$

- $x(n-1) \xleftrightarrow{ZT} z^{-1} X(z) + x(-1)$
- $x(n-2) \xleftrightarrow{ZT} z^{-2} X(z) + z^{-1} x(-1) + x(-2)$
- $x(n+3) \xleftrightarrow{ZT} z^{-3} X(z) + z^{-2} x(-1) + z^{-1} x(-2) + x(-3)$

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2) Time advance property :-

If $x(n) \xleftrightarrow{ZT} X(z)$ then $x(n+k) \xleftrightarrow{z} z^{k}\left[X(z) - \sum_{l=0}^{k-1} x(l) z^{-l}\right]$

Proof :-

$$ZT[x(n+k)] = \sum_{n=0}^{\infty} x(n+k) z^{-n}$$

$n+k = l$

$n = l-k$

$$= \sum_{l=k}^{\infty} x(l) z^{-(l-k)}$$

$$= z^{k} \sum_{l=k}^{\infty} x(l) z^{-l}$$

$$= z^{k}\left[-\sum_{l=0}^{k-1} x(l) z^{-l} + \sum_{l=0}^{k-1} x(l) z^{-l} + \sum_{l=k}^{\infty} x(l) z^{-l}\right]$$

$$= z^{k}\left[-\sum_{l=0}^{k-1} x(l) z^{-l} + \underbrace{\sum_{l=0}^{\infty} x(l) z^{-l}}_{X(z)}\right]$$

$$\boxed{ZT[x(n+k)] = z^{k}\left[X(z) - \sum_{l=0}^{k-1} x(l) z^{-l}\right]}$$

If $k=1 \Rightarrow ZT[x(n+1)] = z[X(z) - x(0)]$

3, Initial value theorem :-

If $x(n) \overset{ZT}{\longleftrightarrow} X(z)$ then $x(0) = \lim_{z \to \infty} X(z)$

Proof :-

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots$$

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots \right]$$

$$= x(0) + 0 + 0 + 0 \dots$$

$$\boxed{x(0) = \lim_{z \to \infty} X(z)}$$

4, Final value theorem :-

If $x(n) \overset{ZT}{\longleftrightarrow} X(z)$ then $x(\infty) = \lim_{z \to 1} (z-1) X(z)$

Proof:-

$$ZT[x(n+1) - x(n)] = \sum_{n=0}^{\infty} [x(n+1) - x(n)] z^{-n}$$

$$ZT[x(n+1) - x(n)] = ZT[x(n+1)] - ZT[x(n)]$$

$$= z[X(z) - x(0)] - X(z)$$

$$= (z-1) X(z) - z\, x(0)$$

$$[z-1] X(z) - z\, x(0) = [x(1) - x(0)] + [x(2) - x(1)] z^{-1} + [x(3) - x(2)] z^{-2} + \dots$$

Apply $\lim_{z\to 1}$ on b.s

$$\lim_{z\to 1}(z-1)X(z) - \lim_{z\to 1} z\,x(0) = \lim_{z\to 1} [x(1)-x(0)]+[x(2)-x(1)]z^{-1} + [x(3)-x(2)]z^{-2}+\dots$$

$$\lim_{z\to 1}(z-1)X(z) - \cancel{x(0)} = x(\infty) - \cancel{x(0)}$$

$$\boxed{\lim_{z\to 1}\left((z-1)X(z)\right) = x(\infty)}$$

this theorem is valid if All the poles of $(z-1)X(z)$ lies inside the unit circle.

Q, Find the final vale of $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$

$$\lim_{z\to 1}(z-1)X(z) = x(\infty) \qquad \frac{z}{z-1/2}$$

Poles $z = 1/2$

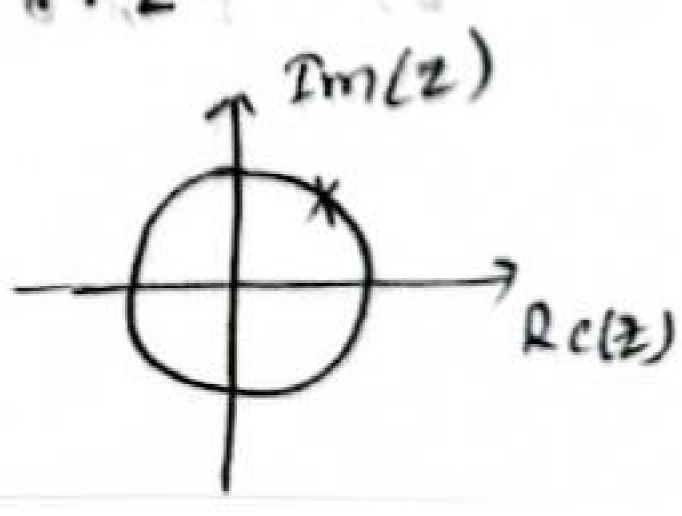
The pole is inside the unit circle

$\therefore$ FVT is applicable

$$\lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}\frac{(z-1)z}{(z-1/2)} = 0$$

Q, Find the final value for $X(z) = \frac{1}{1+z^{-1}}$

$$X(z) = \frac{z}{z+1}$$



Pole $z = -1$

$|z| = 1$

Pole lies on the unit circle $\Rightarrow$ FVT is not applicable

IZT $\Rightarrow$ $x(n) = (-1)^n u(n) \xleftrightarrow{ZT} X(z) = \frac{1}{1+z^{-1}}$

At $n \to \infty$, $x(n)$ oscillates b/w $+1$ & -1

$\therefore$ There is no f.v for the system.

Q, Find the initial & Final value of $X(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1}-0.25z^{-2})}$

<u>FVT</u>

$$\lim_{z \to 1} \frac{\cancel{(z-1)}\, z^3}{\cancel{(z-1)}\,(z^2-0.5z-0.25)} = 4$$

<u>IVT</u>

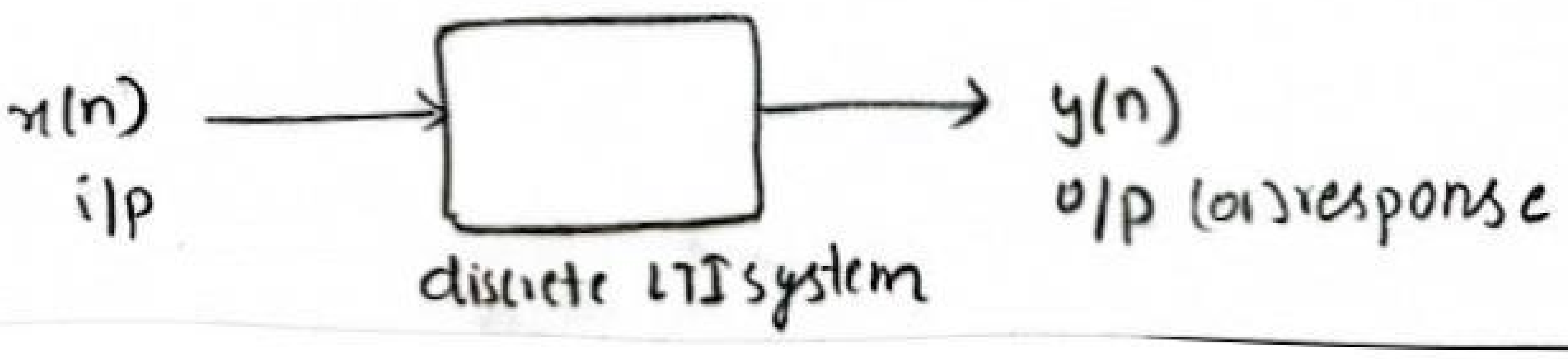
$$x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{1}{(1-z^{-1})(1-0.5z^{-1}-0.25z^{-2})}$$

$$= \frac{1}{(1-0)(1)} = 1$$

Determine the step response of the system

$y(n) = \alpha y(n-1) + x(n)$ where $y(-1) = 1$

$-1 < \alpha < 1$



$\Rightarrow$ Input $x(n) = u(n)$ then find o/p $y(n) = \alpha y(n-1) + x(n)$

$$y(n) = \alpha y(n-1) + x(n)$$

$$z[y(n)] = \alpha z[y(n)-1] + z[x(n)]$$

$$Y(z) = \alpha [z^{-1} Y(z) + y(-1)] + X(z)$$

$$x(n) = u(n) \xleftrightarrow{ZT} X(z) = \frac{1}{1-z^{-1}}$$

$$Y(z) = \alpha [z^{-1} Y(z) + 1] + \frac{1}{1-z^{-1}}$$

$$Y(z)[1-\alpha z^{-1}] = \alpha + \frac{1}{1-z^{-1}}$$

$$Y(z) = \frac{\alpha}{1-\alpha z^{-1}} + \frac{1}{(1-z^{-1})(1-\alpha z^{-1})}$$

$$Y(z) = \frac{\alpha}{1-\alpha z^{-1}} + \frac{1}{1-\alpha}\left[\frac{1}{1-z^{-1}} - \frac{\alpha}{1-\alpha z^{-1}}\right]$$

IZT,

$$W(z) = \frac{1}{(1-z^{-1})(1-\alpha z^{-1})} = \frac{z^2}{(z-1)(z-\alpha)}$$

$$\frac{W(z)}{z} = \frac{z}{(z-1)(z-\alpha)} = \frac{C_1}{z-1} + \frac{C_2}{z-\alpha}$$

$$z = C_1(z-\alpha) + C_2(z-1)$$

If $z = \alpha \Rightarrow \alpha = C_2(\alpha-1) \Rightarrow C_2 = \frac{\alpha}{\alpha-1} = \frac{-\alpha}{1-\alpha}$

If $z = 1 \Rightarrow 1 = C_1(1-\alpha) \Rightarrow C_1 = \dfrac{1}{1-\alpha}$

$$\frac{W(z)}{z} = \frac{1}{1-\alpha}\left[\frac{1}{z-1} - \frac{\alpha}{z-\alpha}\right]$$

$$W(z) = \frac{1}{1-\alpha}\left[\frac{z}{z-1} - \frac{\alpha z}{z-\alpha}\right]$$

$$Y(z) = \frac{\alpha}{1-\alpha z^{-1}} + \frac{1}{1-\alpha}\left[\frac{1}{1-z^{-1}} - \frac{\alpha}{1-\alpha z^{-1}}\right]$$

IZT, $y(n) = \alpha \cdot \alpha^n u(n) + \dfrac{1}{1-\alpha}\left[u(n) - \alpha \cdot \alpha^n u(n)\right]$

$$y(n) = \alpha^{n+1} u(n) + \frac{1}{1-\alpha}\left[u(n) - \alpha^{n+1} u(n)\right]$$

$$\boxed{y(n) = \alpha^{n+1} u(n) + \left[\frac{1-\alpha^{n+1}}{1-\alpha}\right] u(n)}$$